Resilient Networks for Multi-Agent Systems based on Graph Self-Organization into Random Approximate Regular Graphs

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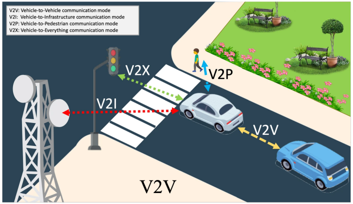
Outline

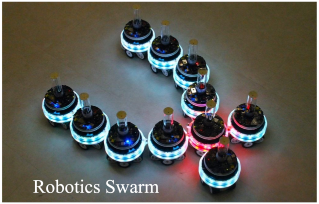
Outline

Background

Multi-agent systems

A **multi-agent system** is composed of multiple units that can interact within a dynamic environment and possess self-organizing capabilities.





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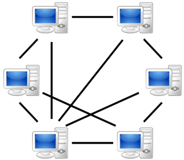

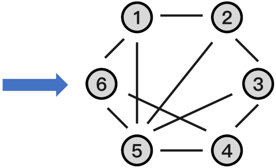

- $\mathcal{G} = (\mathcal{V}, \mathcal{E})$
- $\mathcal{V} = \{1, 2, \ldots, n\}$
- $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$

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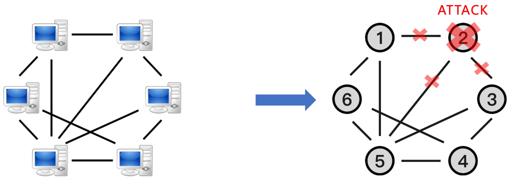


Examples of perturbations:

- Random failures
- Malicious attacks...

CONNECTIVITY

Metrics for measuring Connectivity:

- Node/edge connectivity
- Cost connectivity
- Average Path Length
- Algebraic connectivity (λ_2)...

[1] Bullo F. Lectures on network systems[M]. Santa Barbara, CA: Kindle Direct Publishing, 2019.

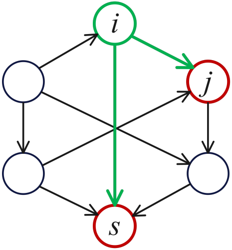
Basic Definitions

Introduction of edge ownership

We consider a scenario in a multi-agent system where the creation of a new edge (connection) between agents grants the creating agent exclusive rights over that edge, modeled by a directed ownership graph $\mathcal{G}_d = (\mathcal{V}, \mathcal{E}_d)$.

Example:

- Agent i creates and own edges (i, j) and (i, s);
- Only edge owners i can remove these two edges;
- $\mathcal{N}_{i,own} = \{v \in \mathcal{V} \mid (i, v) \in \mathcal{E}_d\} = \{j, s\}$;
- The graph G that describes the interconnections among the agents can be viewed as the undirected version of G_d.



$\mathcal{G}_d = (\mathcal{V}, \mathcal{E}_d)$

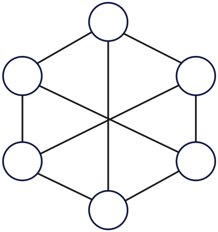
Basic Definitions

Definition of k-regular graph

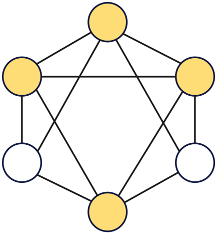
A graph with n nodes is said to be *k-regular* if each node has degree k such that the product nk is even, where $n, k \geq 2$ are integers.

Definition of approximate k-regular graph

A graph with n nodes is said to be *Δ-approximate k-regular* if each node has the degree within $[k, k+\Delta]$, where $n, k \geq 2$ and $\Delta \geq 0$ are integers.



3-regular graph

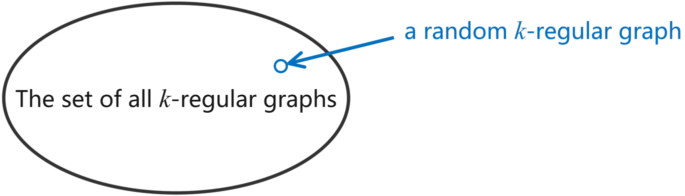


1-approx. 3-regular graph

Basic Definitions

Definition of random k-regular graph

A k-regular graph is said to be *random* if it is selected uniformly at random from all k-regular graphs with the same number of nodes, where $n, k \geq 2$ are integers.



Basic Definitions

Definition of random k-regular graph

A k-regular graph is said to be *random* if it is selected uniformly at random from all k-regular graphs with the same number of nodes, where $n, k \geq 2$ are integers.

Proposition 1 (Algebraic connectivity of random k-regular graph)

Given a random k-regular graph, with high probability (see [4, Theorem 1.1]), the second smallest eigenvalue λ_2 of the Laplacian matrix L_n, the algebraic connectivity, is lower-bounded by

$$\lambda_2 \geq \lambda_{2,lb} := k - 2\sqrt{k-1}.$$

[1] J. Friedman, A proof of alon's second eigenvalue conjecture, in *Proceedings of the thirty-fifth annual ACM symposium on Theory of computing*, 2003.

Problem statement

Problem of interest

We are interested in **increasing network resilience to disconnection**, resulting from the loss of nodes or links, in a distributed manner, leveraging only locally available information.

Outline of the contributions

1. **Design of a distributed protocol** for persistently self-organizing any connected graph into a random 2-approx. k-regular graph with high connectivity, i.e., $\lambda_2 \geq \lambda_{2,lb} := k - 2\sqrt{k-1}$;
2. **Numerical simulations** show that the protocol generates graphs that closely approximate random k-regular graphs in terms of algebraic connectivity and spectral distribution.

Outline

The main difference from the series of works

	Protocol	Node coordination	Connectivity	Proof	Dos attack
[1] Yazıcıoğlu *et.al*	A connected graph with $\overline{d_{init}}$ ↓ A connected random k-regular graph where $k \in [\underline{d_{init}}, \overline{d_{init}}+2]$ with a similar number of edges as initial	✓	✓	✓	✗
conv. faster ↑ [2] Zohreh *et.al*	A connected graph with k ↓ A random k-regular graph possibly "approximate" (only one node without k)	✓	✗	✗	✓
conv. faster ↑ [3] Wenjie *et.al*	A connected graph with k ↓ A random 2-approximate k-regular graph	*"edge ownership"* ✗	✓	✓	✓
	arbitrary choice of the regularity degree		*unlikely to disconnect*	*numerical simulations*	

[1] A. Y. Yazıcıoğlu, M. Egerstedt, and J. S. Shamma, "Formation of robust multi-agent networks through self-organizing random regular graphs, *IEEE Transactions on Network Science and Engineering*, vol. 2, no. 4, 2015.

[2] Z. A. Z. S. Dashti, D. Deplano, C. Seatzu, and M. Franceschelli, "Resilient Self-Organizing Networks in Multi-Agent Systems via Approximate Random k-Regular Graphs", *61st IEEE Conference on Decision and Control*, 2022.

[3] W. Zhao, D. Deplano, Z. Li, A. Giua, M. Franceschelli, "Resilient Networks for Multi-Agent Systems based on Graph Self-Organization into Random Approximate Regular Graphs", *IEEE Conference on Automation Science and Engineering*, 2024.

Outline

Proposed distributed protocol

Protocol 1

Input: A connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, its ownership graph $\mathcal{G}_d = (\mathcal{V}, \mathcal{E}_d)$, and an arbitrary integer degree $k \geq 2$.

Randomizes the graph during infinite execution

- Rule (Add)
- Rule (Remove)
- Rule (Move)

Output: A random 2-approximate k-regular graph with a high algebraic connectivity and a closely spectral distribution as random k-regular graph.

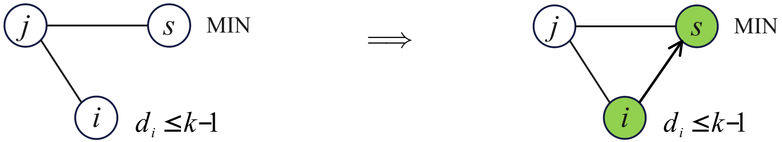
Proposed distributed protocol

Three rules based on local interaction in Protocol 1:

Rule (A): **add edges** while $d_i \leq k-1$ and there is $s \in \mathcal{N}_{ij}^{\mathrm{MIN}}$ where $\mathcal{N}_{ij}^{\mathrm{MIN}} := \{s \in \mathcal{N}_{ij} : d_s = \min_{\ell \in \mathcal{N}_{ij}} d_\ell\}$;

Rule (R): remove edges while $d_i \geq k+1$ and there is $j \in \mathcal{N}_{i,own}$ with the maximum degree such that $d_j \geq k+1$;

Rule (M): try to move or add edges if $\mathcal{N}_{i,own} \neq \emptyset$.



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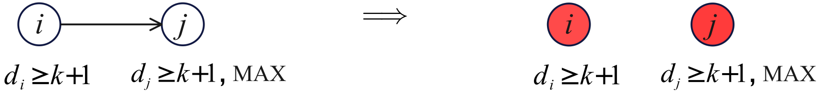

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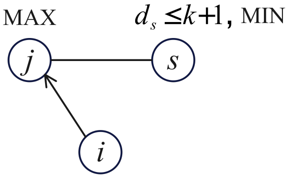

$\Longrightarrow$

Case 1: move an edge if $d_j \geq k+1$

Case 2: add an edge if $d_j < k+1$ and $d_i \leq k+1$

Proposed distributed protocol

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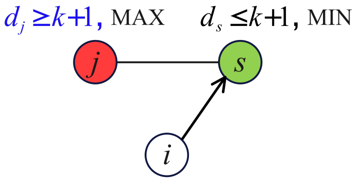
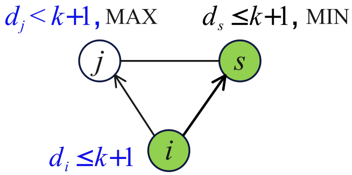
Case 1: move an edge



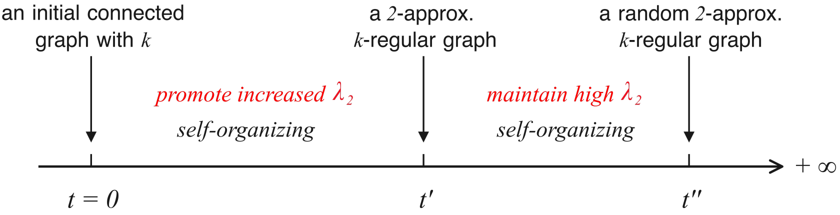
Case 2: add an edge



Theoretical result of the Protocol

Theorem 1

Consider a network of $n \in \mathbb{N}$ agents interacting according to an initial graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and executing the proposed Protocol 1 with parameter $k \geq 2$. If $\mathcal{G}$ is initially connected and remains connected thereafter, then the degree d_i of each agent $i \in \mathcal{V}$ almost surely converges to the interval $d_i \in [k, k+2]$, i.e., $\mathcal{G}$ is reorganized into a 2-approximate k-regular graph.



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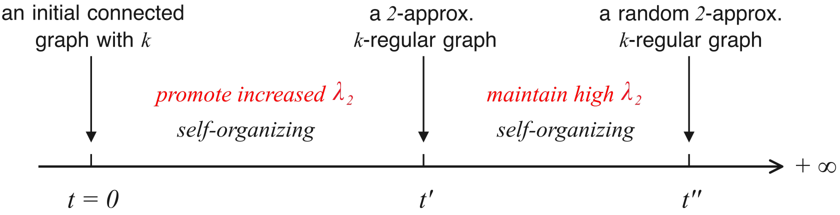


Outline

1 Empirical spectral distribution (ESD)

Proposition 2

Let A_n be the adjacency matrix of a random k-regular graph with n nodes. In the limit of $n \to \infty$, the ESD of the normalized adjacency matrix

$$A_{n,\sigma} = \frac{1}{\sigma} A_n, \qquad \sigma = \sqrt{k-1}$$

approaches the distribution with density

$$\rho_k(x) = \begin{cases} \frac{k^2-k}{2\pi(k^2-kx^2+x^2)}\sqrt{4-x^2} & \text{if } |x| \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

[4] B. D. McKay, The expected eigenvalue distribution of a large regular graph, *Linear Algebra and its applications*, vol. 40, 1981.

[5] H. Kesten, Symmetric random walks on groups, *Transactions of the American Mathematical Society*, vol. 92, no. 2, 1959.

1 Empirical spectral distribution (ESD)

Proposition 3 (Wigner's semicircle law)

Let A_n be the adjacency matrix of a random k-regular graph with n nodes. In the limit of $k, n \to \infty$, the ESD of the normalized adjacency matrix

$$A_{n,\sigma} = \frac{1}{\sigma} A_n, \qquad \sigma = \sqrt{k - k^2/n}$$

approaches the distribution with semicircle density

$$\rho_{sc}(x) = \begin{cases} \frac{1}{2\pi}\sqrt{4 - x^2}, & \text{if } |x| \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

[6] L. V. Tran, V. H. Vu, and K. Wang, Sparse random graphs: Eigenvalues and eigenvectors, *Random Structures & Algorithms*, vol. 42, no. 1, 2013

[7] E. P. Wigner, On the distribution of the roots of certain symmetric matrices, *Annals of Mathematics*, vol. 67, no. 2, 1958.

1 Empirical spectral distribution (ESD)

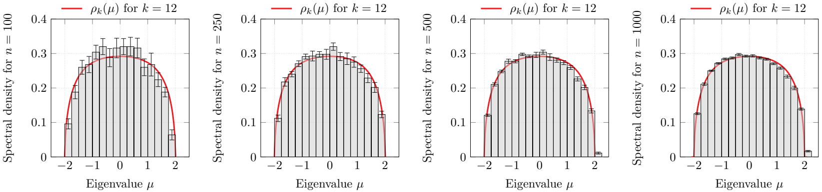


Figure 1: Empirical eigenvalue density histogram of the normalized adjacency matrix $A_n/\sqrt{k-1}$ of graphs generated by Protocol1 in networks with an increasing number of agents $n \in \{100, 250, 500, 1000\}$ and fixed degree of regularity $k = 12$. The red curve represents the density ρ_k expected for large k-regular graphs (see Proposition 2).

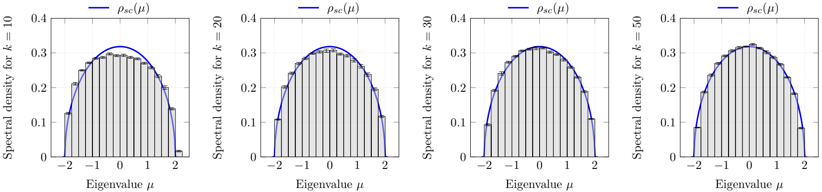


Figure 2: Empirical eigenvalue density histogram of the normalized adjacency matrix $A_n/\sqrt{k - k^2/n}$ of graphs generated by Protocol1 in networks with $n = 1000$ agents and an increasing degree of regularity $k \in \{10, 20, 30, 50\}$. The blue curve represents the semicircle density ρ_{sc} expected for large k-regular graphs as $k \to \infty$ (see Proposition 3).

2 Performance comparison with the state of the art

Example

We consider a network with $n = 1000$ agents initially interacting according to a graph with average degree equal to 10, and we set the desired degree of regularity equal to $k = 50$.

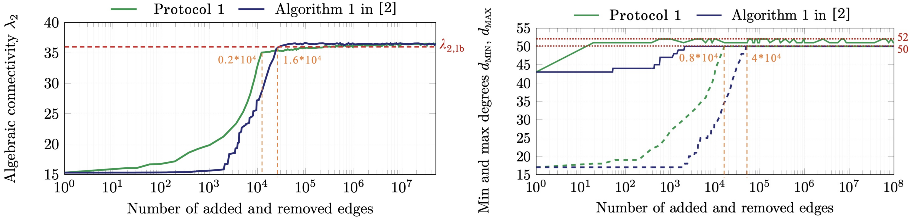


Figure 3: Evolution of λ_2 (left) and the maximum/minimum degrees (right) against the number of added and removed edges during the execution of Protocol 1 (green curves) and Algorithm 1 in [2] (blue curves).

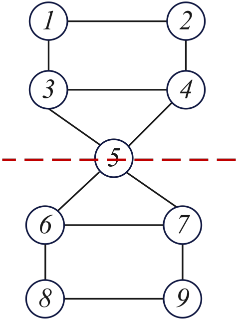
[2] Z. A. Z. S. Dashti, G. Oliva, C. Seatzu, A. Gasparri, and M.bFranceschelli, Distributed mode computation in open multi-agent systems, *IEEE Control Systems Letters*, vol. 6, 2022.

3 Open network under intelligent attack

Fiedler Eigenvector

The eigenvector corresponding to the second smallest eigenvalue (Algebraic Connectivity, $\boldsymbol{\lambda}_2$) of the Laplacian matrix, denoted as $\mathbf{v}_2$.

Example:



$$\boldsymbol{\lambda}_2 = 0.438447, \qquad \mathbf{v}_2 = \begin{bmatrix} -0.429751 \\ -0.429751 \\ -0.268100 \\ -0.268100 \\ 0.000000 \\ 0.268100 \\ 0.268100 \\ 0.429751 \\ 0.429751 \end{bmatrix}$$

3 Open network under intelligent attack

Example

We consider a network with $n = 1000$ agents initially interacting according to a graph with average degree equal to 10, and we set the desired degree of regularity equal to $k = 50$. During the execution of the protocol, agents are selected and removed from the network (*Fiedler vector*), thus simulating DoS attacks.

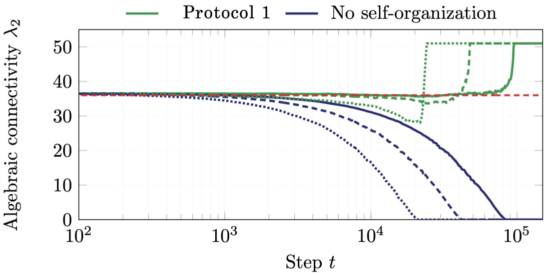


Figure 4: Evolution of λ_2 in initially 50-regular graphs of $n = 1000$ nodes from which one node is disconnected every 100 steps (solid curve), 50 steps (dashed curve), 25 steps (dotted curve).

Outline

Conclusions and future directions

Conclusions:

- A distributed protocol allows for arbitrary regularity degree selection, controlling algebraic connectivity to ensure high connectivity;
- The protocol generates a random 2-approx. k-regular graphs with similar spectral properties to random k-regular graphs, and improving convergence speed without node coordination;
- The protocol maintains good connectivity in open multi-agent systems even under malicious attacks.

Future directions:

- We will formally prove that the protocol maintains connectivity;
- We aim to develop methods for self-organizing r-robust graphs in multi-agent and peer-to-peer networks.

Thank you for your attention

email:
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August 31st, 2024

W. Zhao, D. Deplano, Z. Li, A. Giua, M. Franceschelli, "*Algebraic Connectivity Control and Maintenance in Multi-Agent Networks under Attack*", Automatica, under review. ArXiv:2406.18467.